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Time Reversal Duality Between Linear Networks

David C. Hamill

Abstract—**In time reversal duality, pairs of networks have corresponding waveforms that are time reversed versions of each other. The relation linking a pair of generalized, nonlinear, time varying dynamical systems is derived. It is then applied to time invariant linear networks. Four types of time reversal circuit duality are possible, and in two of these the direction of energy flow is reversed.**

I. Introduction

Time reversal of waveforms has recently been suggested for a variety of signal processing applications. Kobayashi *et al.* [1] applied it to video recording, Ariyavisitakul [2] to combat fading in radio reception, and Hiroi *et al.* [3] to a Hilbert transformation. In all of this work, time reversal was achieved by writing a frame of data into a digital memory, then reading it out in reverse order. The concept of time reversal does not seem to have been applied to analog electronics.

The traditional duality between voltage and current has been well understood for many years, and derives from the topological duality between the nodes and branches of a network graph. The nodal equations of a network have a one-to-one correspondence with the mesh equations of its dual, and vice versa (e.g., see [4], pp. 201–213). Time reversal duality is proposed as the basis of a different relationship, in which the corresponding waveforms of a network and its dual are time reversed versions of each other. The objective of this paper is to develop the theory of time reversal duality as applied to linear networks. In the longer term the theory is expected to be applicable to power converters with bidirectional energy flow.

II. Dynamical Systems

The starting point is an n-dimensional continuous dynamical system D with state vector $\boldsymbol{x} \in \mathbb{R}^n$, characterized by the vector differential equation

$$\frac{d}{dt}\boldsymbol{x}(t) = \boldsymbol{f}[\boldsymbol{x}(t), t] \tag{1}$$

where the derivative function $\boldsymbol{f}$ is an attribute of a particular system. (This normal form description is quite general, and includes systems capable of subharmonics, quasi-periodicity or chaotic behavior.) Given an initial condition $\boldsymbol{x}(0)$, the evolution of the system may be found by integrating (1) to generate a state-space trajectory $\boldsymbol{x}(t)$.

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Let $D^\#$ be another dynamical system with state vector $\boldsymbol{x}^\# \in \mathbb{R}^n$. If $\boldsymbol{x}^\#(t) = \boldsymbol{x}(-t)$ for all t, i.e. if the corresponding waveforms are time reversed versions of each other, then D and $D^\#$ are defined to be time reversal duals.

The relation between $\boldsymbol{f}$ and the corresponding function $\boldsymbol{f}^\#$ that governs $D^\#$ is easily found. Equation (1) remains valid if t is replaced by any other symbol; in particular, it may be replaced by $r = -t$, producing an equivalent description in terms of reverse time

$$\frac{d}{dr}\boldsymbol{x}(r) = \boldsymbol{f}[\boldsymbol{x}(r), r]. \tag{2}$$

Since $dt/dr = -1, d/dr$ in (2) may be replaced by $-d/dt$. Further, from the definitions of $\boldsymbol{x}^\#$ and $r, \boldsymbol{x}(r) = \boldsymbol{x}^\#(t)$. With these substitutions, (2) becomes

$$\begin{aligned}\frac{d}{dt}\boldsymbol{x}^\#(t) &= -\boldsymbol{f}[\boldsymbol{x}^\#(t), -t]\\ &= \boldsymbol{f}^\#[\boldsymbol{x}^\#(t), t]\end{aligned} \tag{3}$$

which is the differential equation governing $D^\#$. As $\boldsymbol{x}^\#(0) = \boldsymbol{x}(0)$, the two systems clearly have identical initial conditions. The transformation from $\boldsymbol{f}$ to $\boldsymbol{f}^\#$ involves no more than negation of $\boldsymbol{f}$ and t, so $D^\#$ always exists and is unique.

III. Linear Electrical Networks

How may this result be applied to synthesize the time reversal dual of an electrical network? More precisely, given a network N, how may a dual network $N^\#$ be constructed, having a topology similar to N but time reversed waveforms, i.e., voltages $v^\#(t) = v(-t)$ and currents $i^\#(t) = i(-t)$? The task is not as straightforward as might be supposed, and it turns out that there is more than one way to accomplish it.

For simplicity, the electrical networks considered here contain only the following elements:

1) linear, time invariant inductance L, capacitance C, and resistance R;
2) independent, time-varying sources of current $J(t)$ and voltage $E(t)$.

(The symbols should be understood to denote generalized inductance, capacitance, etc., rather than particular elements in the network.) Other circuit elements may be handled in a similar way to these.

A network may be described by the constitutive relations of its elements and the constraints imposed on its currents and voltages (Kirchhoff's laws). If the individual constitutive relations and constraints are correctly transformed, then the whole network N will be properly transformed into its time reversal dual $N^\#$.

First the constraints: KCL and KVL are valid for any t, and are therefore invariant under time reversal. Next, consider the constitutive relations which link the branch currents and voltages. Elements L and C are governed by differential equations, and are therefore transformed according to (3). If N contains an element L satisfying $di/dt = v/L$, then $N^\#$ must contain an element $L^\#$ satisfying $di^\#/dt = -v^\#/L^\#$. Similarly, for capacitance $dv/dt = i/C$ transforms to $dv^\#/dt = -i^\#/C^\#$. The constitutive relation of R is algebraic and holds for any t, so it is invariant under time reversal: $v = Ri$ transforms to $v^\# = R^\# i^\#$. Sources $J(t)$ and $E(t)$ are also described algebraically, but depend explicitly upon time. This dependency must be reversed: $i = J(t)$ transforms to $i^\# = J^\#(t) = J(-t); v = E(t)$ transforms to $v^\# = E^\#(t) = E(-t)$.

TABLE I
TYPES OF TIME REVERSAL CIRCUIT DUALITY

	$i^{\#}(t)$	$v^{\#}(t)$	$L^{\#}$	$C^{\#}$	$R^{\#}$	$J^{\#}(t)$	$E^{\#}(t)$
Type 1:	$-i(-t)$	$v(-t)$	L	C	$-R$	$-J(-t)$	$E(-t)$
Type 2:	$i(-t)$	$-v(-t)$	L	C	$-R$	$J(-t)$	$-E(-t)$
Type 3:	$i(-t)$	$v(-t)$	$-L$	$-C$	R	$J(-t)$	$E(-t)$
Type 4:	$-i(-t)$	$-v(-t)$	$-L$	$-C$	R	$-J(-t)$	$-E(-t)$

It is clear that certain changes of sign (negation) are involved in the transformation of $\{i, v, L, C, R, J, E\}$ to $\{i^{\#}, v^{\#}, L^{\#}, C^{\#}, R^{\#}, J^{\#}, E^{\#}\}$. Note that the transformations for L and C involve negation, whereas those for the other elements do not. Moreover, the negations must be consistent with Kirchhoff's laws. KCL holds if *all* currents are negated, and KVL holds if *all* voltages are negated.

There are four ways to satisfy the inductance transformation:

1) negate i, leaving v and L unchanged;
or 2) negate v, leaving i and L unchanged;
or 3) negate L, leaving i and v unchanged;
or 4) negate i, v and L.

Introducing the Boolean variable $\overline{B_x}$ for "negate x" (and B_x for "leave x unchanged"), the inductance transformation T_L may be expressed as

$$T_L = \overline{B_i} B_v B_L + B_i \overline{B_v} B_L + B_i B_v \overline{B_L} + \overline{B_i}\,\overline{B_v}\,\overline{B_L}. \quad (4)$$

Similar expressions may be developed for the other circuit element transformations. The overall transformation is then $T = T_L T_C T_R T_J T_E$, which simplifies to the sum-of-products form

$$\begin{aligned} T = {}& \underbrace{\overline{B_i} B_v B_L B_C \overline{B_R}\,\overline{B_J}\, B_E}_{\text{Type 1}} + \underbrace{B_i \overline{B_v} B_L B_C \overline{B_R} B_J \overline{B_E}}_{\text{Type 2}} \\ & + \underbrace{B_i B_v \overline{B_L}\,\overline{B_C} B_R B_J B_E}_{\text{Type 3}} + \underbrace{\overline{B_i}\,\overline{B_v}\,\overline{B_L}\,\overline{B_C} B_R \overline{B_J}\,\overline{B_E}}_{\text{Type 4}} \quad (5) \end{aligned}$$

Thus it turns out that there are four ways of forming the time reversal circuit dual, named here Types 1–4. They are summarized in Table I.

IV. DISCUSSION

A. The Four Types of Duality

In Type 1 time reversal duality, the required negation of i to form $i^{\#}$ can be achieved by inverting the reference direction of all currents, including current sources. Type 1 is perhaps the easiest of the four to visualize. Imagine an animated motion picture showing the operation of a circuit with its currents represented by moving charges. If the movie were projected in reverse, the charges would appear to move backwards, while the voltages would remain in their original polarity. All waveforms would be time reversed.

For Type 2 the reference polarity of all voltages, including voltage sources, is inverted. For Type 3, L and C in the original network are replaced by $L^{\#} = -L$ and $C^{\#} = -C$ in the dual.

Type 1 and Type 2 duals are themselves related by a further transformation, in which every current and every voltage is inverted. (Such a transformation, though almost trivial, is the basis by which a circuit including npn bipolar transistors may be turned into one for pnp transistors, for example.) Types 3 and 4 are related by the same transformation. Thus it is arguable that there are fundamentally just two types of time reversal circuit duality: one in which resistance is negated (Type 1–2), and another in which reactance is negated (Type 3–4). Though negative L, C and R are physically unrealizable, their characteristics may be emulated with the aid of a negative impedance converter (NIC).

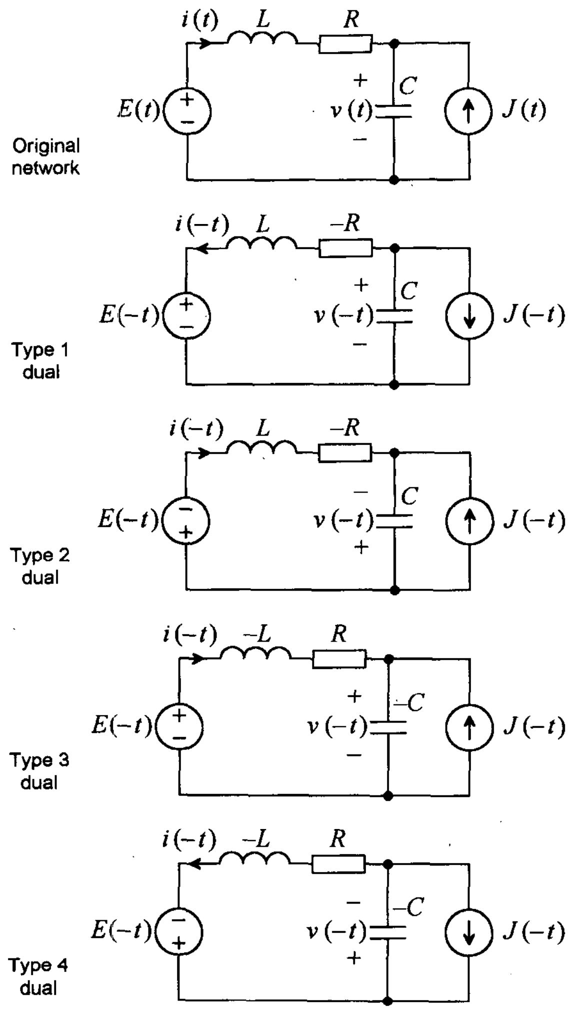

Fig. 1. A network and its four time reversal duals.

Fig. 1 shows an example of a simple network and its four time reversal duals.

B. Combination with Voltage–Current Duality

If N is planar, the voltage-current duality transformation may be applied to create a network N^* in which $v^*(t) = i(t)$, etc. This network may subsequently be transformed by time reversal duality to produce a network $(N^*)^{\#} = N^{*\#}$ in which $v^{*\#}(t) = i(-t)$, etc. It does not matter in which order the transformations are applied; the operations are commutative, $(N^{\#})^* \equiv (N^*)^{\#}$.

C. Frequency Domain

The frequency domain properties of time reversal are also of interest. From the definition of the Fourier transform $\mathcal{F}$ it is easily shown that

$$\mathcal{F}[x(t)] = X(j\omega) \Longleftrightarrow X^{\#}(j\omega) = \mathcal{F}[x(-t)] = -X(-j\omega) \quad (6)$$

Similarly, it follows from the definition of the Laplace transform $\mathcal{L}$ that

$$\mathcal{L}[x(t)] = X(s) \Longleftrightarrow X^{\#}(s) = \mathcal{L}[x(-t)] = -X(-s). \quad (7)$$

An important consequence of (7) is that if a linear circuit or system is asymptotically stable, i.e., if all its poles lie in the left half plane,

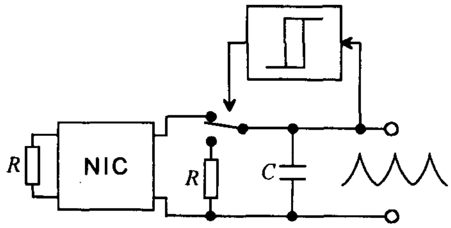

Fig. 2. Exponential signal generator based on Type 1 time reversal duality.

then its time reversal dual will be unstable, and vice versa. Though an unstable circuit may appear to be useless, this is not necessarily so: with a bounded input, the circuit's output will also be bounded *over a finite time interval.* This property has been put to use in the exponential signal generator shown in Fig. 2, which makes use of an NIC to emulate negative resistance for Type 1 duality. The output voltage alternately grows and decays between limits set by a hysteretic comparator.

D. Reversal of Energy Flow

In Type 1–2 time reversal duality, either current or voltage is negated, but not both. Power $p^{\#}$ in the dual circuit is given by

$$p^{\#}(t) = v^{\#}(t)i^{\#}(t) = -v(-t)i(-t) = -p(-t). \tag{8}$$

Because the power waveforms throughout the dual network are negated, the direction of energy flow is reversed. This is an important consideration for circuits in the field of power electronics, because it allows the synthesis of power converters with reversed or bidirectional energy flow. Such converters have many practical applications [5]–[7]. Type 1–2 duality can be applied, because ideal power converters are lossless, containing no dissipative R elements but only L, C and ideal switching devices. If an ideal power converter is considered realizable, then its Type 1–2 dual is also realizable.

To proceed further with such circuits, the theory of time reversal must be extended to include switches and diodes. Difficult philosophical questions arise with switches, because they are controlled devices and the direction of causality is reversed under time reversal. Work on clarifying this issue is currently in progress.

V. Conclusion

The time reversal dual of a network may be formed in four different ways. An important property of Type 1 and Type 2 duals is that the direction of energy flow is reversed. With extension to include switching devices, the theory of time reversal duality should find application in power electronics.

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Comment on "The Single CC II Biquads with High-Input Impedance"

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In the above paper[1], two configurations for realizing second-order voltage transfer functions using a single current conveyor (CC II) were given. There is an error in the basic equation of the first configuration, which is (4). The correct form of (4) is:

$$\frac{V_1}{V_y} = \frac{y_2 y_5 - y_1 y_4}{y_5(y_2 + y_3 + y_4) + y_3 y_4}. \tag{4}$$

The sign error in (4) resulted in a series of sign errors in (9), (13) (first equation), (14), (15), and (19).

The filter networks obtained from the second configuration have one more resistor R_5 which has no effect on the filter transfer functions as seen from (16) and (17), thus taking $R_5 = 0$, the two networks described in 1 and 2 of Table III are identical to those obtained from configuration 1. The third network of Table III employs three capaitors to realize band- and high-pass responses with real axis poles, thus it is of no value.

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