

Circuits and Systems Letters

Subharmonics and Chaos in a Controlled Switched-Mode Power Converter

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Abstract—A difference equation is derived for the output current, at successive switching events, of a simple switching-regulator dc/dc converter employing a pulsewidth modulator (PWM). Wideband feedback control of the nonlinear circuit leads to a one-dimensional return map of zigzag form from which a stability criterion is found. Operation in the unstable region is described, and verified by numerical simulation. The system exhibits a “noisy” bifurcation, chaos and subharmonics.

I. INTRODUCTION

Chaotic dynamics have been observed and described in a variety of simple nonlinear electronic circuits [1]–[7]. With a few exceptions [6], [7] these are not circuits that find everyday application; but it has been shown [8], [9] that chaos can occur in nonlinear feedback systems, notably those containing a pulsewidth modulator (PWM). Many power electronic circuits in common use fall into this category. This letter describes in detail the operation of such a circuit, a switched-mode power converter, in its chaotic region.

II. ANALYSIS

We consider the switching-regulator dc/dc converter circuit (buck converter) of Fig. 1. All components are assumed to be ideal and lossless. For simplicity it is assumed that the current i in the choke is always nonzero; that is, operation is confined to the usual continuous-current mode. However, this restriction may easily be removed as shown below. The load is a constant-voltage sink. The switching frequency is taken as constant with period T and the switch operates with a duty factor D , $D \in [0, 1]$.

The choke current at the end of the n th cycle of operation is i_n . The switch is then closed for a time DT during which the current rises linearly to $i_n + (V_I - V_O)DT/L$. The switch is opened for the remainder of the cycle during which the diode conducts and the current falls linearly by an amount $V_O(1 - D)T/L$. A difference equation may be written relating the currents at the end of two consecutive cycles:

$$i_{n+1} = i_n + (V_I D - V_O)T/L. \tag{1}$$

Closed-loop control of the choke current is by means of a PWM driven by a current-error signal e_n sampled at the end of each cycle:

$$e_n = A(I_{ref} - i_n) \tag{2}$$

where I_{ref} is a notional reference current and A is a gain having appropriate dimensions. The PWM duty factor for the $(n + 1)$ th cycle is determined by e_n , and is constrained to lie in the interval

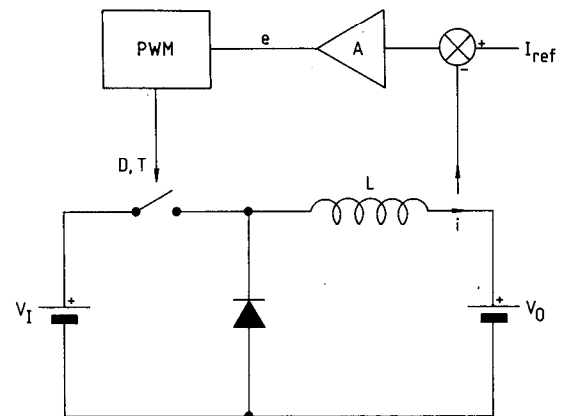


Fig. 1. Switching-regulator dc/dc converter with wideband feedback control.

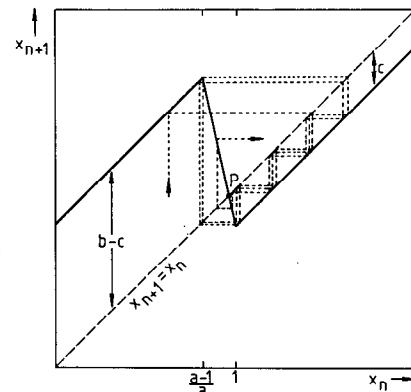


Fig. 2. The zigzag return map for the controlled switching regulator, drawn for large a and b/c close to 5. The geometrical construction illustrates how an intermittent period-5 subharmonic can evolve.

$[0, 1]$ by a nonlinear saturation function $\text{sat}(\cdot)$:

$$D = \text{sat}(e_n) = \begin{cases} 0, & e_n \leq 0 \\ e_n, & 0 < e_n < 1 \\ 1, & e_n \geq 1. \end{cases} \tag{3a}$$

$$\tag{3b}$$

$$\tag{3c}$$

The difference equation for the closed-loop case is obtained from (1), (2) and (3) in a normalized form as

$$x_{n+1} = x_n + b \text{sat}[a(1 - x_n)] - c \tag{4}$$

where $x_{n+1} = i_{n+1}/I_{ref}$, $x_n = i_n/I_{ref}$, $a = AI_{ref}$, $b = V_I T/(I_{ref} L)$, and $c = V_O T/(I_{ref} L)$; these are all positive, dimensionless quantities.

The difference equation (4) may be represented graphically by a one-dimensional return map. (We note an early use of the return map to study PWM control system stability [10] that predates the recognition of chaotic dynamics.) The return map for this circuit is the piecewise-linear folded mapping illustrated in Fig. 2, whose form suggests that sobriquet “zigzag map.” Its three linear segments correspond to the three regions of (3) and

Manuscript received July 29, 1987.
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IEEE Log Number 8719098.

expressions may be obtained for each:

$$x_{n+1} = \begin{cases} x_n + b - c, & x_n \leq (a-1)/a & (5a) \\ x_n(1-ab) + ab - c, & (a-1)/a < x_n < 1 & (5b) \\ x_n - c, & x_n \geq 1. & (5c) \end{cases}$$

(However, if the circuit is allowed to operate in the discontinuous-current mode, a fourth segment must be added, $x_{n+1} = 0$, to model the effect of the diode in preventing negative current.)

The map intersects the line of unit slope $x_{n+1} = x_n$ at a single fixed point P in segment (5b) provided that $b > c$ and $a > 0$. These conditions are satisfied under normal operation of the switching regulator. For a small perturbation from P , the sequence $\{x_n\}$ converges to the fixed point (and the system is therefore stable) provided that the absolute value of the gradient of the map at P is less than unity [11], [12].

Stable Region: The gradient of the zigzag return map at P is found from (5b) as $1 - ab$. The fixed point is stable if and only if $|1 - ab| < 1$. Thus for stability, $0 < a < 2/b$.

Unstable Region: When $a \leq 0$ no finite fixed point exists, since the map is not intersected by the line of unit slope. When $a = 2/b$ a bifurcation in the state-space trajectory occurs. The extrema of the zigzag map are two vertices where a discontinuity of slope occurs. Inspection of the r th return map for $a = 2/b$ and $r = 2, 4, 6, \dots$ reveals that all even-order maps contain an identical segment that lies on the line of unit slope, $x_{n+r} = x_n$. Then for $a > 2/b$ each gives rise to intermittency. The effect is that of an apparently noisy period-2 subharmonic, the "noise" worsening as a is increased. The motion is aperiodic and may be considered chaotic in the sense that (4) has periodic solutions of every even integer period.

Some interesting properties are revealed by simple geometrical constructions around the return map. After any initial transient has decayed x_n is bounded by the vertices of the zigzag map: $\{x_n\} \in [1 - c, 1 + b - c - 1/a]$. When $a \geq 3/b$, a period-3 solution appears, implying chaos [13] in the strict sense that (4) then has periodic solutions of every integer period. In the special case when b/c is equal to an integer $k > 1$, any chaotic region terminates at $a = 1/c$, where a period- k attractor emerges and persists as a is increased without limit. However, in practice the ratio V_1/V_0 cannot be set exactly (because of experimental uncertainty) so b/c will just be very close to k . Then a period- k subharmonic appears intermittently—the mechanism is shown in Fig. 2.

III. SIMULATION

The difference equation (4) was investigated by numerical iteration for a variety of parameter values. A representative case is illustrated by Fig. 3, an iteration diagram computed for b/c close to 5 with a as the control parameter. The stable region, "noisy" bifurcation, chaotic region and intermittent period-5 subharmonic are in agreement with our analysis.

The discontinuity of slope at the extrema of the zigzag map may be avoided by altering the saturation function (3). For instance, if segment (3b) is replaced by the cubic spline $\text{sat}(e_n) = 3e_n^2 - 2e_n^3$, $0 < e_n < 1$, the map has smooth, quadratic extrema where the derivatives are all finite and the lowest non-vanishing derivative is the second. Thus a classical Feigenbaum route to chaos is followed [11], [12]. The iteration diagram now contains pitchfork bifurcations, bands of chaos and stable subharmonic windows, as shown in Fig. 4. In the case that the lowest derivative that does not vanish at the extrema is the m th, the chaotic bands become sparser and the stable subharmonic windows become wider with increasing m [14].

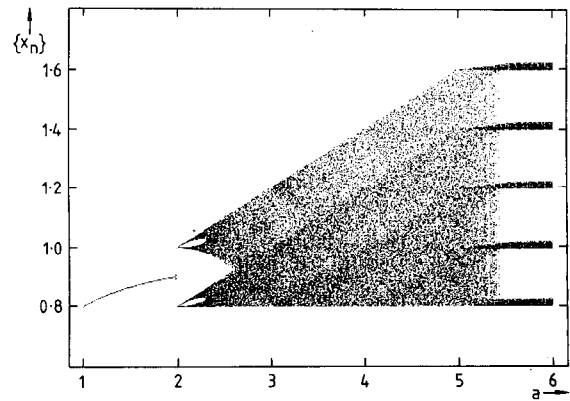


Fig. 3. Iteration diagram for the case $b=1$, $c=0.201$, using the piecewise-linear saturation function (3). $\{x_n\}$ is plotted for $n > 256$ against the control parameter a .

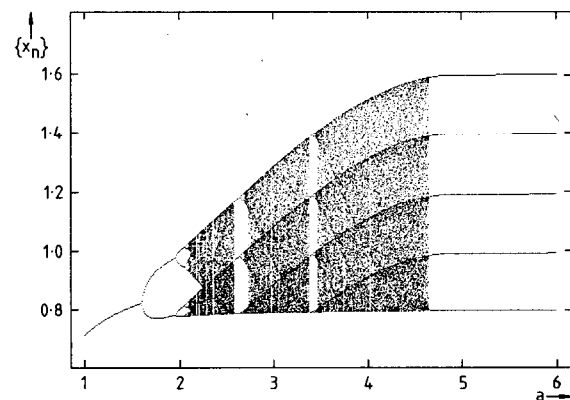


Fig. 4. As Fig. 3, but with segment (3b) of the saturation function replaced by the cubic spline $\text{sat}(e_n) = 3e_n^2 - 2e_n^3$, $0 < e_n < 1$. Wide subharmonic windows of period 2, 3, 4 and 5 are present. The intervening bands of chaos are reached by a period-doubling route.

IV. CONCLUSION

In common with many other chaotic systems, the circuit studied is conceptually simple, essentially comprising a PWM and an integrator. Although the dynamics are complex, the parameters at which subharmonics and chaos commence can be predicted accurately, as can the bounds of the chaotic current excursions, and the frequencies emitted from the power supply. These factors are important in engineering power supplies to operate within quantifiable design limits.

The circuit described and its related derivatives are widely used. Our findings may be directly extended to cover buck-derived circuits such as the forward converter, and similar principles may be applied to other power converters. Extension to higher order circuits is not straightforward; the return map becomes of higher dimension and many more exotic features will, therefore, appear [12].

REFERENCES

- [1] J. Testa, J. Perez, and C. Jeffries, "Evidence for universal chaotic behaviour of a driven nonlinear oscillator," *Phys. Rev. Lett.*, vol. 48, pp. 714-717, Mar. 1982.
- [2] E. Freire, L. G. Franquelo, and J. Aracil, "Periodicity and chaos in an autonomous electronic system," *IEEE Trans. Circuits Syst.*, vol. CAS-31, pp. 237-247, Mar. 1984.
- [3] T. Matsumoto, L. O. Chua, and S. Tanaka, "Simplest chaotic nonautonomous circuit," *Phys. Rev. A*, vol. 30, pp. 1155-1157, Aug. 1984.
- [4] T. Matsumoto, L. O. Chua, and K. Tokumasu, "Double scroll via a two-transistor circuit," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 828-835, Aug. 1986.

- [5] M. P. Kennedy and L. O. Chua, "Van der Pol and chaos," *IEEE Trans. Circuits Syst.*, vol. CAS-33, pp. 974-980, Oct. 1986.
- [6] D. J. Jefferies, "Bifurcation to chaos in clocked digital systems containing autonomous timing circuits," *Phys. Lett. A*, vol. 115, pp. 89-92, Mar. 1986.
- [7] J. H. B. Deane and D. J. Jefferies, "Periodic, intermittent and chaotic states of autonomous digital systems," 1987, to be published.
- [8] J. Baillieul, R. W. Brockett, R. B. Washburn, "Chaotic motion in nonlinear feedback systems," *IEEE Trans. Circuits Syst.*, vol. CAS-27, pp. 990-997, Nov. 1980.
- [9] A. Mees, "Chaos in feedback systems," in *Chaos*, A. Holden (Ed.), Manchester, England: Manchester Univ. Press, 1986, pp. 99-110.
- [10] E. Polak, "Stability and graphical analysis of first-order pulse-width-modulated sampled-data regulator systems," *IRE Trans. Auto. Contr.*, vol. AC-6, pp. 276-282, September 1961.
- [11] M. J. Feigenbaum, "Universal behaviour in nonlinear systems," *Los Alamos Science*, vol. 1, pp. 4-27, 1980 (reprinted in [12]).
- [12] P. Cvitanovic (Ed.), "Universality in chaos," Adam Hilger Ltd., Bristol, England, 1984.
- [13] T. Y. Li and J. A. Yorke, "Period three implies chaos," *Am. Math. Monthly*, vol. 82, no. 10, pp. 985-992, 1975.
- [14] D. J. Jefferies, unpublished.

On the Application of the No-Gain Property in Nonlinear Resistive Networks

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Abstract—The necessary and sufficient topological conditions under which one can transfer all independent sources of a no-gain nonlinear resistive network to the branches with positive linear resistors are derived. Such a relocation of sources allows to estimate, practically without computation, the greatest possible voltage and current in the circuit.

I. INTRODUCTION

In this paper let \mathcal{N} denote any consistent circuit made of independent nonzero voltage and current sources, linear passive resistors, 2-terminal, \dots , n -terminal nonlinear resistors, short-circuit elements which do not form loops by themselves and with voltage sources only and open-circuit elements which do not form cutsets by themselves and with current sources only. If all nonlinear resistors satisfy the no-gain criterion, then \mathcal{N} has the no-gain property which essentially says that the magnitude of the voltages between any pair of nodes (currents flowing into each terminal of every element) is less than or equal to the sum of the magnitudes of the voltages (currents) appearing across (flowing through) the independent sources [1], [2]. The correct values of these sums are very difficult to establish when both kinds of independent sources occur together in the circuit because the circuit must usually be solved to get the voltages across (currents through) the current (voltage) sources. On the other hand, they provide the initial bounds on the solutions of the circuit and are useful in computer simulation. In certain cases, it is possible to redistribute the independent sources without changing the circuit's operating point so that each of them, being associated with a positive linear resistor, can be simultaneously represented in its Thévenin and Norton forms which immediately enable us to simply calculate the bounds on any voltage or current in the circuit.

The aim of this note is to present necessary and sufficient conditions on the topology of the network under which such a

redistribution, understood in the sense of operations defined in Theorem 1, can be performed. The proof of the main proposition involves a corollary derived from Minty's Colored Arc Theorem [3] and is one more example of applications of this powerful result.

II. VASCHY'S THEOREM AND COLORED BRANCH COROLLARY

The way in which the distribution of independent sources will be changed is explained below. This is Vaschy's well-known theorem on equivalent circuits.

Theorem 1: Let \mathcal{C} (\mathcal{L}) be an arbitrary cutset (loop) of \mathcal{N} determined by an independent voltage (current) source $E(I)$ and oriented in the same direction as $E(I)$. Let \mathcal{N}^* be the circuit derived from \mathcal{N} by connecting a source identical to $E(I)$ directed opposite to \mathcal{C} (\mathcal{L}) in series (parallel) with every branch of \mathcal{C} (\mathcal{L}). Then \mathcal{N} and \mathcal{N}^* have the same operating points.

The proof of this property results from the observation that in any connected graph an oriented cutset and loop have an even number of common branches, half being of the same direction and half directed opposite in the cutset and loop ([4, Theorems 2.14 and 6.5]). This implies that the above operations do not violate the equations describing the circuit so that its operating point remains unaltered. It is easily seen that these operations consist in setting the source $E(I)$ in an indicated branch to zero and associating it with all other branches of a chosen cutset (loop).

The second theorem recalled here is a corollary derived in [3].

Theorem 2: Let b be any branch of a graph. Label it the dark green branch and color each of the remaining branches arbitrarily in red or in blue. Then branch b either forms a loop exclusively with red branches, or a cutset exclusively with blue branches, but not both.

III. MAIN RESULT

To prove the main theorem we introduce an additional terminology; namely, we will say that a loop (cut set) of \mathcal{N} containing a number of voltage (current) sources is nonzero if the sum of their voltage (current) magnitudes, each one measured with respect to the picked orientation of the loop (cutset), is nonzero.

Theorem 3: All independent voltage (current) sources of the network \mathcal{N} can, only by applying the operations specified in Theorem 1, be associated with its branches containing linear resistors if, and only if, there are in \mathcal{N} no nonzero loops (cutsets) formed exclusively by nonlinear resistors, short (open)-circuits and voltage (current) sources.

Proof:

(if part): It is impossible to remove all independent voltage (current) sources from a nonzero loop (cutset) formed exclusively by nonlinear resistors short (open)-circuits and voltage (current) sources by using only the operations defined in Theorem 1 because they do not alter the above specified sum of the voltages (currents) so that this loop (cutset) always remains nonzero.

(only if part): We prove only the voltage sources case. Suppose there are in \mathcal{N} no nonzero loops formed exclusively by nonlinear resistors, short-circuits and voltage sources. We prove by induction on the number k of voltage sources being not in series with positive linear resistors that all they can be transferred to such branches. This is trivially satisfied when $k = 0$, so let $k \geq 1$ and assume the truth for all \mathcal{N} of the number of "single" voltage

Manuscript received September 23, 1987. This paper was supported by the National Program for Fundamental Research 02.14.

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IEEE Log Number 8820126.